

Geometrical method for thermal instability of nonlinearly charged BTZ Black Holes

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In this paper we consider three dimensional BTZ black holes with three models of nonlinear electrodynamics as source. Calculating heat capacity, we study the stability and phase transitions of these black holes. We show that Maxwell, logarithmic and exponential theories yield only type one phase transition which is related to the root(s) of heat capacity. Whereas for correction form of nonlinear electrodynamics, heat capacity contains two roots and one divergence point. Next, we use geometrical approach for studying classical thermodynamical behavior of the system. We show that Weinhold and Ruppeiner metrics fail to provide fruitful results and the consequences of the Quevedo approach are not completely matched to the heat capacity results. Then, we employ a new metric for solving this problem. We show that this approach is successful and all divergencies of its Ricci scalar and phase transition points coincide. We also show that there is no phase transition for uncharged BTZ black holes.

I. INTRODUCTION

One of the interesting subjects for recent gravitational studies is the investigation of three dimensional black holes [1–9]. Considering three dimensional solutions helps us to find a profound insight in the black hole physics, quantum view of gravity and also its relations to string theory [10–14]. Moreover, three dimensional spacetimes play an essential role to improve our understanding of gravitational interaction in low dimensional manifolds [15]. Due to these facts, some of physicists have an interest in the $(2 + 1)$ -dimensional manifolds and their attractive properties [16–27].

The Maxwell theory is in agreement with experimental results, but it fails regarding some important issues such as self energy of point-like charges which motivates one to regard nonlinear electrodynamics (NED). There are some evidences that motivate one to consider NED theories: solving the problem of point-like charge self energy, understanding the nature of different complex systems, obtaining more information and insight regarding to quantum gravity, compatible with AdS/CFT correspondence and string theory frames, description of pair creation for Hawking radiation and the behavior of the compact astrophysical objects such as neutron stars and pulsars [28–30]. Therefore, many authors investigated the black hole solutions with nonlinear sources [31–49].

On the other hand, thermodynamical structure of the black holes, has been of great interest. It is due to the fact that, according to AdS/CFT correspondence, black hole thermodynamics provides a machinery to map a solution in AdS spacetime to a conformal field on the boundary of this spacetime [13–15]. Also, it was recently pointed out that considering cosmological constant as a thermodynamical variable leads to the behavior similar to the Van Der Waals liquid/gas system [50–57]. In addition, phase transition of the black holes plays an important role in exploring the critical behavior of the system near critical points. There are several approaches that one can employ to study the phase transition. One of these approaches is studying the behavior of the heat capacity. It is argued that roots and divergence points of the heat capacity are representing two types of phase transition [58–64]. In addition, studying the heat capacity and its behavior, enable one to study the thermal stability of the black holes [65–70].

Another approach for studying phase transition of black holes is through thermodynamical geometry. The concept is to construct a spacetime by employing the thermodynamical properties of the system. Then, by studying the divergence points of thermodynamical Ricci scalar of the metric, one can investigate phase transition points. In other words, it is expected that divergencies of thermodynamical Ricci scalar (TRS) coincide with phase transition points of the black holes. Firstly, Weinhold introduced differential geometric concepts into ordinary thermodynamics [71, 72]. He considered a kind of metric defined as the second derivatives of internal energy with respect to entropy and other extensive quantities for a thermodynamical system. Later Ruppeiner [73, 74] introduced another metric and defined the minus second derivatives of entropy with respect to the internal energy and other extensive quantities. The Ruppeiner metric is conformal to the Weinhold metric with the inverse temperature as the conformal factor. It is notable that, both metrics have been applied to study the thermodynamical geometry of ordinary systems [75–80]. In particular, it was found that the Ruppeiner geometry carries information of phase structure of thermodynamical system. For the systems with no statistical mechanical interactions (for example, ideal gas), the scalar curvature

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is zero and the Ruppeiner metric is flat. Because of the success of their applications to ordinary thermodynamical systems, they have also been used to study black hole phase structures and lots of results have been obtained for different types of black holes [81–86].

It is notable that these two approaches fail in order to describe phase transition of several black holes [87]. In order to overcome this problem, Quevedo proposed new types of metrics for studying geometrical structure of the black hole thermodynamics [88, 89]. This method was employed to study the geometrical structure of the phase transition of the black holes [58–64, 90–94] and proved to be a strong machinery for describing phase transition of the black holes. But this approach was not completely coincided with the results of classical thermodynamics arisen from the heat capacity [87]. In Ref. [87], a new metric was proposed in which the denominator of its Ricci scalar is only constructed of numerator and denominator of the heat capacity. Several phase transition of the black holes have been studied in context of the HPEM (Hendi-Panahiyan-Eslam Panah-Momennia) metric [87].

In this paper we study thermal stability and phase transition of the BTZ black holes in presence of several NED models in context of heat capacity. Then, we employ Weinhold, Ruppeiner and Quevedo methods for studying geometrothermodynamics of these black holes. We will see that Weinhold and Ruppeiner metrics fail to provide fruitful results and the consequences of the Quevedo approach are not completely matched to the heat capacity results. Then, we employ the HPEM metric and study the phase transition of these black holes in context of geometrothermodynamics. We end the paper with some closing remarks.

II. NONLINEARLY CHARGED BTZ BLACK HOLE SOLUTIONS

The $(2+1)$ -dimensional action of Einstein gravity with NED field in the presence of cosmological constant is given by

$$I = -\frac{1}{16\pi} \int d^3x \sqrt{-g} [R - 2\Lambda + L(\mathcal{F})], \quad (1)$$

where R is the Ricci scalar, the cosmological constant is $\Lambda = -l^{-2}$ in which l is a scale factor. Also, $L(F)$ is the Lagrangian of NED, in which we consider three models. First model was proposed by Hendi (HNED) [95], second one is Soleng theory (SNED) [96] and third one is correction form of NED (CNED) [69]

$$L(\mathcal{F}) = \begin{cases} \beta^2 \left(\exp\left(-\frac{\mathcal{F}}{\beta^2}\right) - 1 \right), & \text{HNED} \\ -8\beta^2 \ln\left(1 + \frac{\mathcal{F}}{8\beta^2}\right), & \text{SNED} \\ -\mathcal{F} + \alpha\mathcal{F}^2 + \mathcal{O}(\alpha^2), & \text{CNED} \end{cases}, \quad (2)$$

where β and α are called the nonlinearity parameters, the Maxwell invariant $\mathcal{F} = F_{\mu\nu}F^{\mu\nu}$ in which $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field tensor and A_μ is the gauge potential. We should note that for $\beta \rightarrow \infty$ (HNED and SNED branches) and $\alpha \rightarrow 0$ (CNED branch) the Maxwell Lagrangian can be recovered.

The nonlinearly charged static black hole solutions can be introduced with the following line element

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\theta^2, \quad (3)$$

where the metric function $f(r)$ was obtained in Refs. [70, 95]

$$f(r) = \frac{r^2}{l^2} - m + \begin{cases} \frac{\beta r q (1-2L_W)}{\sqrt{L_W}} - \frac{\beta^2 r^2}{2} + q^2 \left[\ln\left(\frac{\beta^2 l^2}{2q^2}\right) - Ei\left(1, \frac{L_W}{2}\right) - \gamma + 3 \right], & \text{HNED} \\ 4\beta^2 r^2 \left[\ln\left(\frac{\Gamma+1}{2}\right) + 3 \right] - q^2 \left[\ln\left(\frac{\beta^2 r^4 (\Gamma-1)(\Gamma+1)^3}{4q^2 l^2}\right) + \frac{6}{\Gamma-1} - 2 \right], & \text{SNED} \\ -2q^2 \ln\left(\frac{r}{l}\right) - \frac{2\alpha q^4}{r^2} + \mathcal{O}(\alpha^2), & \text{CNED} \end{cases}, \quad (4)$$

where $\Gamma = \sqrt{1 + \frac{q^2}{r^2 \beta^2}}$, m and q are integrations constant which are related to mass parameter and the electric charge of the black hole, respectively. In addition, $L_W = \text{LambertW}\left(\frac{4q^2}{\beta^2 r^2}\right)$, $\gamma = \gamma(0) \simeq 0.57722$ and the special function $Ei(1, x) = \int_1^\infty \frac{e^{-xz}}{z} dz$.

The entropy and the electric charge of the obtained NED black hole solutions can be calculated with the following forms [70, 95]

$$S = \frac{\pi r_+}{2}, \quad (5)$$

$$Q = \frac{\pi q}{2}, \quad (6)$$

where r_+ is the event horizon of black hole. On the other hand, the quasi-local mass, which is related to geometrical mass, can be obtained as [70, 95]

$$M = \frac{\pi}{8}m. \quad (7)$$

Regarding Eqs. (5) and (6) with (7) and obtaining m by using $f(r = r_+) = 0$, for these three cases of BTZ black holes we can write

$$M = \frac{1}{8\pi} \times \begin{cases} \frac{4\beta S Q(1-2L'_W)}{\sqrt{L'_W}} - \frac{2S^2(l^2\beta^2-2)}{l^2} + 4Q^2 \left[\ln \left(\frac{\beta^2 l^2 \pi^2}{Q^2} \right) - Ei \left(1, \frac{L'_W}{2} \right) - \gamma + 3 \right], & HNED \\ \frac{4S^2}{l^2} + 16\beta^2 S^2 \left[\ln \left(\frac{\Gamma'+1}{2} \right) + 3 \right] - 4Q^2 \left[\ln \left(\frac{\beta^2 S^4 (\Gamma'-1)(\Gamma'+1)^3}{Q^2 l^2 \pi^2} \right) + \frac{6}{\Gamma'-1} - 2 \right], & SNED \\ \frac{4S^2}{l^2} - 8Q^2 \ln \left(\frac{2S}{l\pi} \right) - \frac{8Q^4}{S^2} \alpha, & CNED \end{cases}, \quad (8)$$

where $\Gamma' = \sqrt{1 + \frac{Q^2}{S^2\beta^2}}$ and $L'_W = LambertW \left(\frac{4Q^2}{\beta^2 S^2} \right)$. Having conserved and thermodynamic quantities at hand, it was shown that the first law of thermodynamics may be satisfied [95]. The main goal of this paper is investigating thermal stability and phase transition for these black holes.

A. Heat capacity

In order to investigate thermal stability and phase transition, one can usually adopt two different approaches to the matter at hand. In one method, the electric charge is considered as a fixed parameter and heat capacity of the black hole will be calculated. The positivity of the heat capacity is sufficient to ensure the local thermal stability of the solutions and its divergencies are corresponding to the phase transition points. This approach is known as canonical ensemble. Another approach for studying thermal stability of the black holes is grand canonical ensemble. In this approach, thermal stability is investigated by calculating the determinant of Hessian matrix of $M(S, Q)$ with respect to its extensive variables. The positivity of this determinant also represents the local stability of the solutions. Although these two approaches are different fundamentally, one expects that the results being the same for both ensembles; i.e. ensemble independent. Here we use the first method for studying thermal stability. For this purpose, the system is considered to be in fixed charge and the heat capacity has the following form

$$C_Q = \left(\frac{\partial M}{\partial S} \right)_Q \left(\frac{\partial^2 M}{\partial S^2} \right)_Q^{-1}. \quad (9)$$

It is notable that, when we study heat capacity for investigating the phase transition, we encounter with two different phenomena. In one, the changes in the signature of the heat capacity is representing a phase transition of the system. In other words, if the heat capacity is negative, then the system is in thermally unstable phase, whereas for the case of the positive C_Q , the system is thermally stable. The roots of the heat capacity in this case are representing phase transition points which means one should solve the following equation

$$\left(\frac{\partial M}{\partial S} \right)_Q = 0, \quad (10)$$

where hereafter we call this type of the phase transition as type one. It is a matter of calculation to show that

$$\left(\frac{\partial M}{\partial S}\right)_Q = \begin{cases} -\frac{2l^2 Q^2 e^{\frac{1}{2}L'_W} \sqrt{L'_W} + S[S(\beta^2 l^2 - 2)\sqrt{L'_W}(1+L'_W) + Q\beta l^2 L'_W(2L'_W - 1) + 2Q\beta l^2]}{2\pi S l^2 \sqrt{L'_W}(1+L'_W)}, & HNED \\ \frac{4\beta^2 l^2 Q^2 (Q^2 - S^2 \beta^2) \ln\left(\frac{1+\Gamma'}{2}\right) + 8S^2 Q^2 l^2 \beta^4 + Q^4 (1+S^2 + 8\beta^2 l^2)}{2\pi S^3 \beta^4 l^2 \Gamma' (1-\Gamma') (1-\Gamma'^2)} + \\ \frac{S^4 \beta^2 (1-S^2 \sqrt{\Gamma'}) - 6Q^4 l^2}{2\pi S^3 \beta^2 l^2 (1-\Gamma') (1-\Gamma'^2)} + \frac{2S^2 \ln\left(\frac{\Gamma'+1}{2}\right) + 6\beta^2 + Q^2}{\pi S (1-\Gamma')}, & SNED \\ \frac{(2Q^4 \alpha - S^2 Q^2) l^2 + S^4}{\pi l^2 S^2}, & CNED \end{cases}, \quad (11)$$

The other case of phase transition is the divergency of the heat capacity. In other words, the singular points of the heat capacity are representing places in which system goes under phase transition. This assumption leads to the fact that the roots of the denominator of the heat capacity are representing phase transitions. Therefore, we have the following relation for this type of phase transition

$$\left(\frac{\partial^2 M}{\partial S^2}\right)_Q = 0, \quad (12)$$

where we call this type of the phase transition as type two. Due to economical reasons, we did not write the explicit relations of $\left(\frac{\partial^2 M}{\partial S^2}\right)_Q$ for different BTZ solutions.

B. Geometrical Thermodynamics

In order to have an effective geometrical approach for studying phase transition of a system, one can build a suitable thermodynamical metric and investigate its Ricci scalar. Thermodynamical metrics were introduced based on the Hessian matrix of the mass (internal energy) with respect to the extensive variables. Therefore, although the electric charge is a fixed parameter for calculating the heat capacity in canonical ensemble, it may be an extensive variable for constructing thermodynamical metrics. In this method we expect that TRS diverges in both types of the mentioned phase transition points. In other words, the denominator of TRS must be constructed in a way that contains roots of the denominator and numerator of the heat capacity. In what follows, we will study the denominator of TRS of the several geometrical approaches and follow the recently proposed thermodynamical metric which its denominator only contains numerator and denominator of the heat capacity, and therefore, divergencies of TRS coincide with roots and divergencies of the heat capacity.

In order to find the roots and divergence points of heat capacity, we should solve its numerator and denominator, separately. Solving the mentioned equations with respect to entropy, leads to

$$S_0 \equiv S|_{C_Q=0} = \begin{cases} \frac{4Q\beta l^2 L\varpi}{(2-\beta^2 l^2)\sqrt{1+2L\varpi}}, & HNED \\ \frac{Q}{\beta\sqrt{L\omega(2+L\omega)}}, & SNED \\ \frac{Q}{2}\sqrt{\pm 2l(\sqrt{l^2 - 8\alpha} \pm l)}, & CNED \end{cases} \quad (13)$$

and

$$S_\infty \equiv S|_{C_Q \rightarrow \infty} = \begin{cases} \frac{2Q\beta l^2}{(\beta^2 l^2 - 2)\sqrt{2\ln\left(1 - \frac{2}{\beta^2 l^2}\right)}}, & HNED \\ \frac{Q}{2\beta\sqrt{\exp\left(-\frac{1}{4\beta^2 l^2}\right)\left[\exp\left(-\frac{1}{4\beta^2 l^2}\right) - 1\right]}}, & SNED \\ \frac{Q}{2}\sqrt{2l(\sqrt{l^2 + 24\alpha} - l)}, & CNED \end{cases} \quad (14)$$

where

$$L\varpi = \text{Lambert}W\left[-\frac{(\beta^2 l^2 - 2)}{2\beta^2 l^2 e^{\frac{1}{2}}}\right], \quad (15)$$

$$L\omega = \text{Lambert}W \left[-2 \exp \left(-\frac{8\beta^2 l^2 + 1}{4\beta^2 l^2} \right) \right]. \quad (16)$$

It is evident from obtained equation for HNED and SNED that there is only one real positive entropy in which heat capacity vanishes. Interestingly, in case of CNED theory we find two roots for heat capacity. It is evident that the roots are increasing functions of the electric charge in these theories. As for nonlinearity parameter, in case of the HNED and SNED theories, the root is an increasing function of β . Whereas for CNED theory, the smaller root is an increasing function of α whereas the larger root is a decreasing function of it.

Now we are in position to study the existence of the type two phase transition point which is related to divergency of the heat capacity. Considering HNED branch of Eq. (14), one finds S_∞ is not real for all values of l and β . Therefore, there is no physical divergence point for the heat capacity of HNED model. Next, for the case of SNED model the same behavior is seen. In other words, since $0 \leq \exp(\frac{-1}{x}) < 1$ for $0 \leq x < \infty$, we can not obtain real S_∞ for all values of l and β . Next, we should investigate CNED model. Regarding Eq. (14), we find that there is a divergence point for heat capacity (we should note that in this paper we consider positive α). In other words, this theory of nonlinear electromagnetic field enjoys the phase transition of type two. The divergence point is an increasing function of the nonlinearity and electric charge parameters. It is worthwhile to mention that for the case of vanishing nonlinearity parameter in this theory, S_∞ goes to zero. In other words, there is no divergence point for heat capacity of Maxwell theory. It is also clear that in case of chargeless BTZ black holes, there is no root and divergence point for heat capacity. In other words, in case of chargeless BTZ black holes, there is not any kind of phase transition.

1. Weinhold and Ruppeiner metrics

The Weinhold metric was given in [71, 72]

$$dS_W^2 = g_{ab}^W dX^a dX^b, \quad (17)$$

where $g_{ab}^W = \partial^2 M(X^c) / \partial X^a \partial X^b$ and also $X^a \equiv X^a(S, N^i)$, where N^i denotes other extensive variables of the system. In case of Weinhold approach, one is considering the mass of the system as potential, other parameters such as entropy and electric charge as extensive parameters and related quantities such as temperature and electric potential as intensive parameters.

The Ruppeiner metric was defined as [73, 74]

$$dS_R^2 = g_{ab}^R dX^a dX^b, \quad (18)$$

where $g_{ab}^R = -\partial^2 S(X^c) / \partial X^a \partial X^b$ and $X^a \equiv X^a(M, N^i)$. In this case the thermodynamical potential is entropy. It is worthwhile to mention that according to the proposal of the Quevedo, these two approaches are related to each other by a Legendre transformation [97].

Taking into account thermodynamical metrics of Weinhold and Ruppeiner, one can obtain their Ricci scalars. Since we would like to investigate divergence points of TRS, \mathcal{R} , we focus on its denominator ($D\mathcal{R}$). One finds

$$D\mathcal{R}^{Win} = M^2 (M_{SS} M_{QQ} - M_{SQ}^2)^2, \quad (19)$$

$$D\mathcal{R}^{Rup} = T M^2 (M_{SS} M_{QQ} - M_{SQ}^2)^2, \quad (20)$$

where $M_k = \frac{\partial M}{\partial k}$ and $M_{kj} = \frac{\partial^2 M}{\partial k \partial j}$.

2. The Quevedo metrics

The Quevedo metrics have two kinds with the following forms [88, 89]

$$dS_R^2 = g_{ab}^Q dX^a dX^b, \quad (21)$$

where g_{ab}^Q is

$$g_{ab}^Q = \Upsilon \begin{pmatrix} -M_{SS} & 0 \\ 0 & M_{QQ} \end{pmatrix}, \quad (22)$$

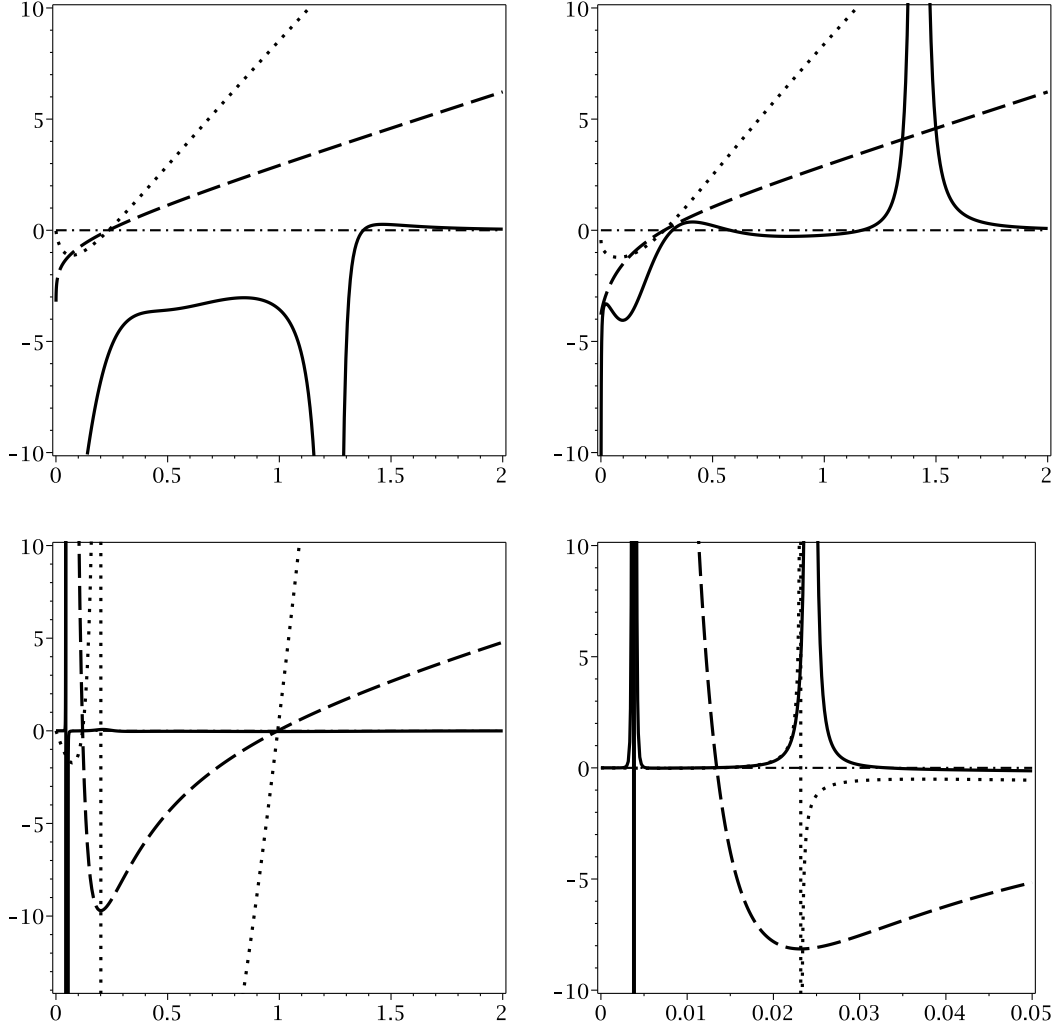


FIG. 1: Weinhold Ricci scalar (solid line), heat capacity (dotted line) and temperature (dashed line) versus S for $l = 1$. **HNED model:** (up-left panel) and **SNED model:** (up-right panel): $q = 0.3$, $\beta = 1$. **CNED model:** (down-left panel) and **CNED model:** (down-right panel): $q = 1$, $\alpha = 0.007$ (different scales).

with

$$\Upsilon = \begin{cases} SM_S + QM_Q, & \text{Case I} \\ SM_S, & \text{Case II} \end{cases}.$$

Taking into account Quevedo metrics, one can find that their related (denominator of) Ricci scalars can be written as

$$D\mathcal{R}^{Q-I} = M_{SS}^2 M_{QQ}^2 (SM_S + QM_Q)^3, \quad (23)$$

$$D\mathcal{R}^{Q-II} = S^3 M_S^3 M_{SS}^2 M_{QQ}^2, \quad (24)$$

3. HPEM metric

In order to avoid any extra divergencies in TRS which may not coincide with phase transitions of the type one and two, and also ensure that all the divergencies of the TRS coincide with phase transition points of the both types, HPEM metric was introduced [87]

$$g_{ab} = S \frac{M_S}{M_{QQ}^3} \begin{pmatrix} -M_{SS} & 0 \\ 0 & M_{QQ} \end{pmatrix}. \quad (25)$$

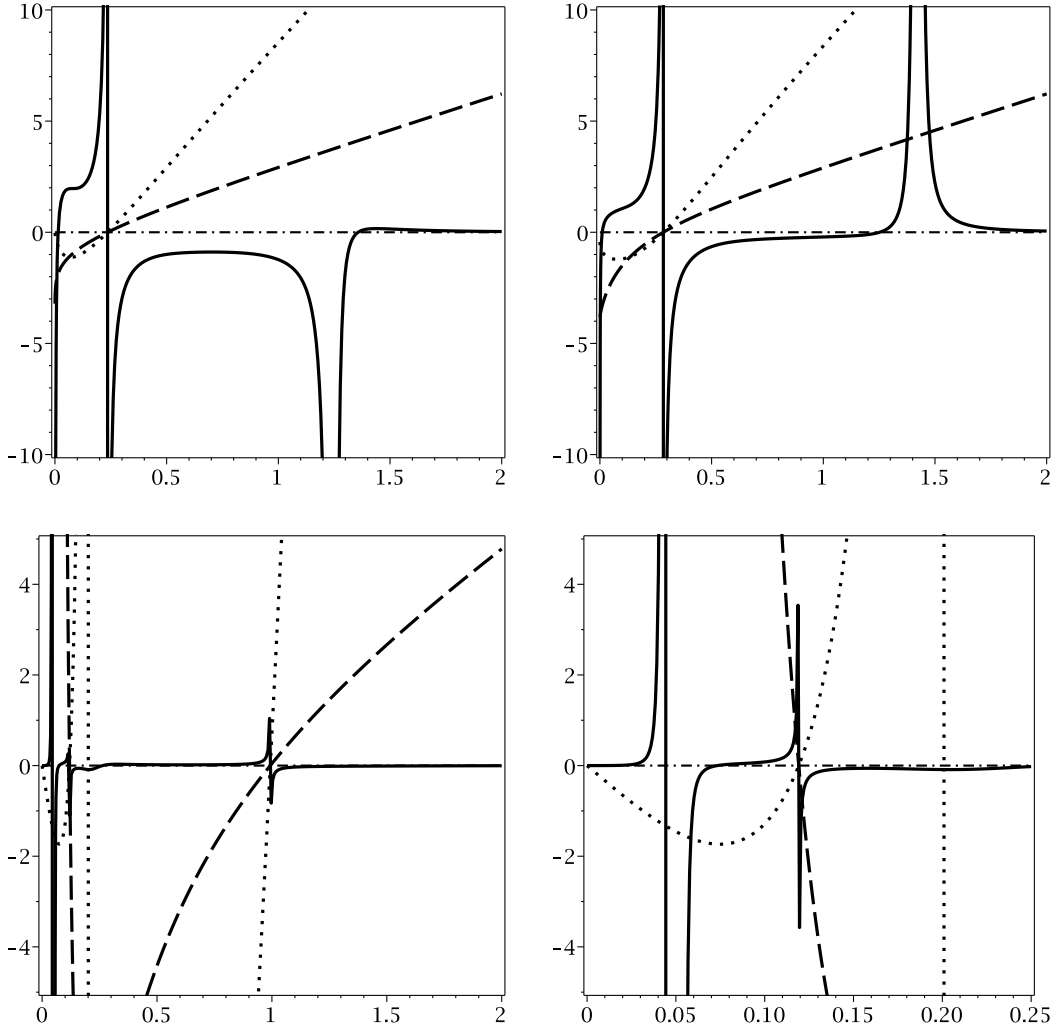


FIG. 2: Ruppeiner Ricci scalar (solid line), heat capacity (dotted line) and temperature (dashed line) versus S for $l = 1$. **HNED model:** (up-left panel) and **SNED model:** (up-right panel): $q = 0.3, \beta = 1$. **CNED model:** (down-left panel) and **CNED model:** (down-right panel): $q = 1, \alpha = 0.007$ (different scales).

In this case we have considered the total mass as thermodynamical potential, entropy and electric charge as extensive parameters. Calculations show that denominator of TRS leads to

$$D\mathcal{R}^{HPEM} = S^3 M_S^3 M_{SS}^2. \quad (26)$$

III. THE RESULTS OF VARIOUS APPROACHES

Here, we investigate phase transitions of black holes using geometrothermodynamics. For this purpose, we used thermodynamical metrics introduced in previous section for the black holes solutions obtained in the section II.

For Weinhold metric, none of divergencies of the Ricci scalar coincide with roots of the heat capacity in every theories of NED models that we have considered in this paper. On the other hand, one of the divergencies of TRS and divergence point of the heat capacity in CNED theory, coincide with each other. It is notable that, in cases of the HNED and SNED theories, there is one divergence point for TRS (up panels of Figs. 1) whereas for CNED, there are two divergencies (down panels of Figs. 1).

In case of Ruppeiner metric, for HNED and SNED models, there is a root for heat capacity in which Ricci scalar of the Ruppeiner metric has a divergence. But there is also another divergence point for Ricci scalar which does not coincide with any phase transition point (up panels of Figs. 2). Therefore, there is an extra divergence point. In

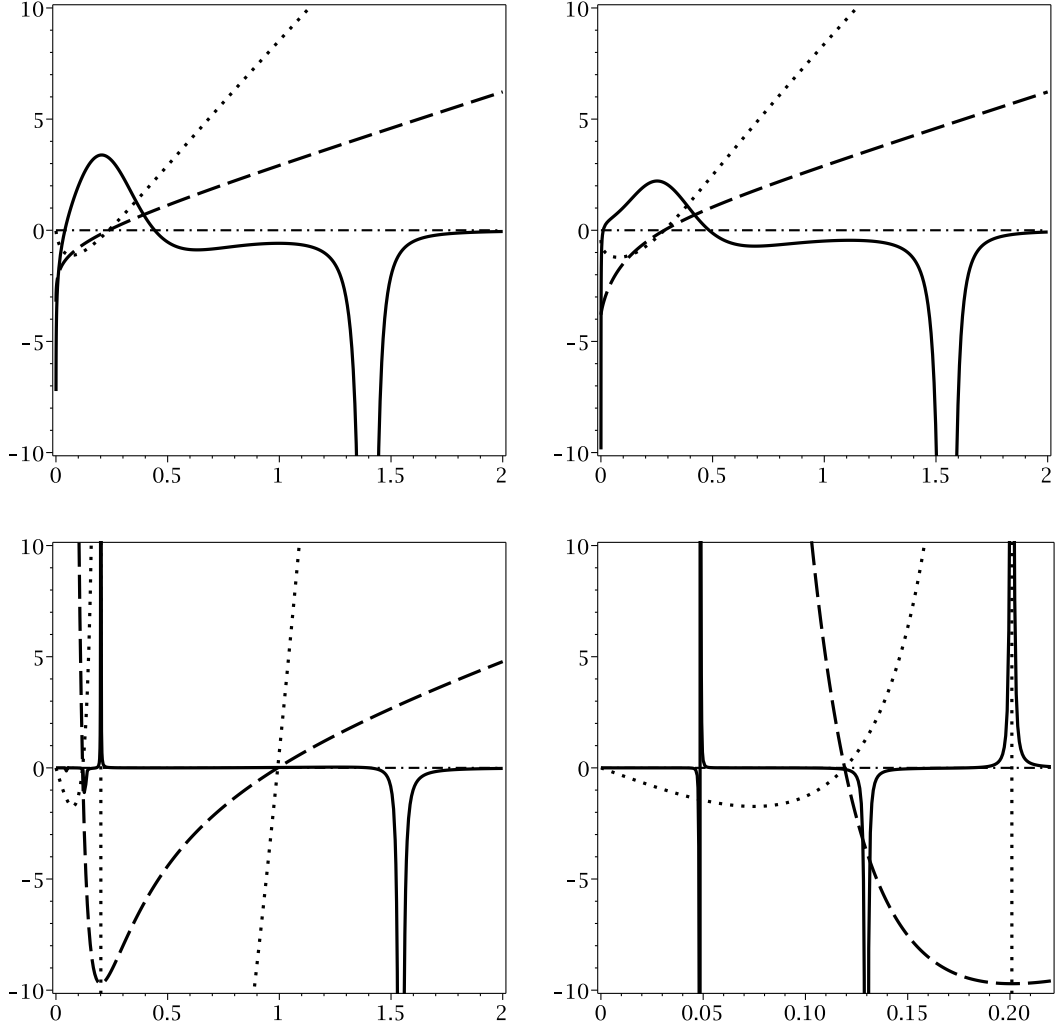


FIG. 3: Quevedo Ricci scalar case I (solid line), heat capacity (dotted line) and temperature (dashed line) versus S for $l = 1$. **HNED model:** (up-left panel) and **SNED model:** (up-right panel): $q = 0.3$, $\beta = 1$. **CNED model:** (down-left panel) and **CNED model:** (down-right panel): $q = 1$, $\alpha = 0.007$ (different scales).

case of the CNED model, two roots and one divergence point are observed for heat capacity in which Ricci scalar has related divergencies. In addition to these divergence points, one extra divergence point is also observed which is not related to any phase transition point (down panels of Figs. 2).

As for Quevedo metrics, for case I, similar behavior as Weinhold is observed for all three theories of NED (Fig. 3). On the other hand, in case of the other metric of Quevedo, two divergence points for Ricci scalar were observed for HNED and SNED theories. One of these divergence points coincides with root of the heat capacity for these two nonlinear theories whereas the other one does not (up panels of Figs. 4). In case of CNED theory, all the divergence points of TRS coincide with phase transition points except one (down panels of Figs. 4). In other words, Quevedo's metric predicts an extra divergence point, corresponding to the equation $M_{QQ} = 0$, which is not predicted in classical black hole thermodynamics.

It is evident that in case of HPEM, all types of phase transition points of heat capacity coincide with divergencies of TRS of HPEM method (Figs. 5). In other words, independent of the nonlinear theory under consideration, the HPEM method provide a machinery in which no extra divergency for TRS is observed and divergence points of TRS and phase transition points coincide. Another interesting and important property of the HPEM method is the behavior of TRS near divergence point for different types of phase transition. As one can see, the signature and behavior of TRS near divergence point for phase transition type one and two are different. Therefore, independent of studying the heat capacity, one can distinguish these types of phase transition from one another only by studying the behavior

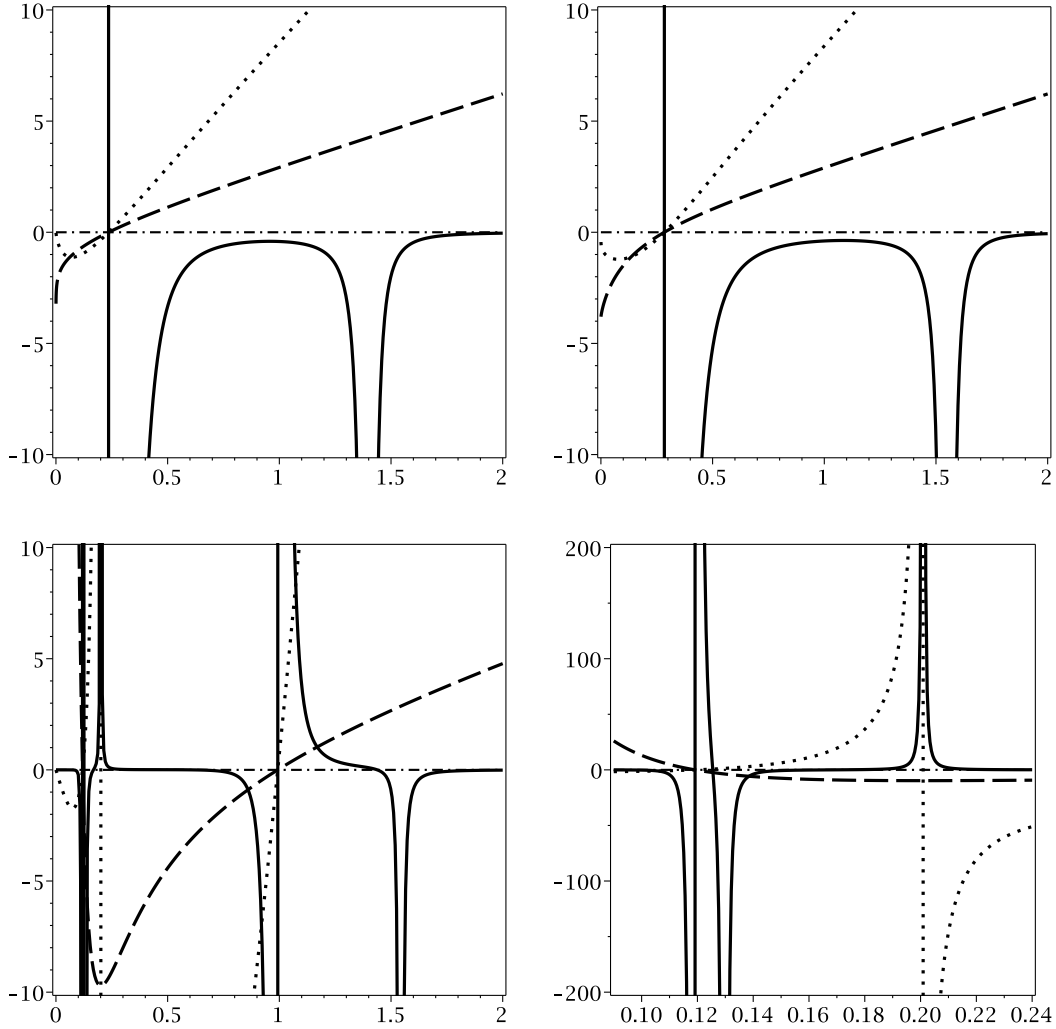


FIG. 4: Quevedo Ricci scalar case II (solid line), heat capacity (dotted line) and temperature (dashed line) versus S for $l = 1$. **HNED model:** (up-left panel) and **SNED model:** (up-right panel): $q = 0.3$, $\beta = 1$. **CNED model:** (down-left panel) and **CNED model:** (down-right panel): $q = 1$, $\alpha = 0.007$ (different scales).

of TRS.

IV. CLOSING REMARKS

In this paper, we have considered BTZ black holes, in presence of three models of NED. We studied stability and phase transitions related to the heat capacity of the mentioned black holes. Next, we employed geometrical approach to study the thermodynamical behavior of the system. In other words, we have studied phase transitions of the system through Weinhold, Ruppeiner and Quevedo methods. Also, we used the recently proposed approach to study geometrical thermodynamics.

We found that the Weinhold and Ruppeiner metrics for studying these BTZ solutions fail to provide a suitable result. In addition, the divergence points of the Quevedo TRS were not completely matched with the phase transition points of the heat capacity results. In other words, in these approaches, the existence of extra divergencies were observed which were not related to any phase transition point in the classical thermodynamics. In some of these approaches no divergency of TRS coincided with phase transition points. In order to obtain a consistent results with the classical thermodynamic consequences (the heat capacity), we employed a new thermodynamical metric. In this approach, all the divergencies of TRS coincided with phase transition points. In other words, roots and divergence

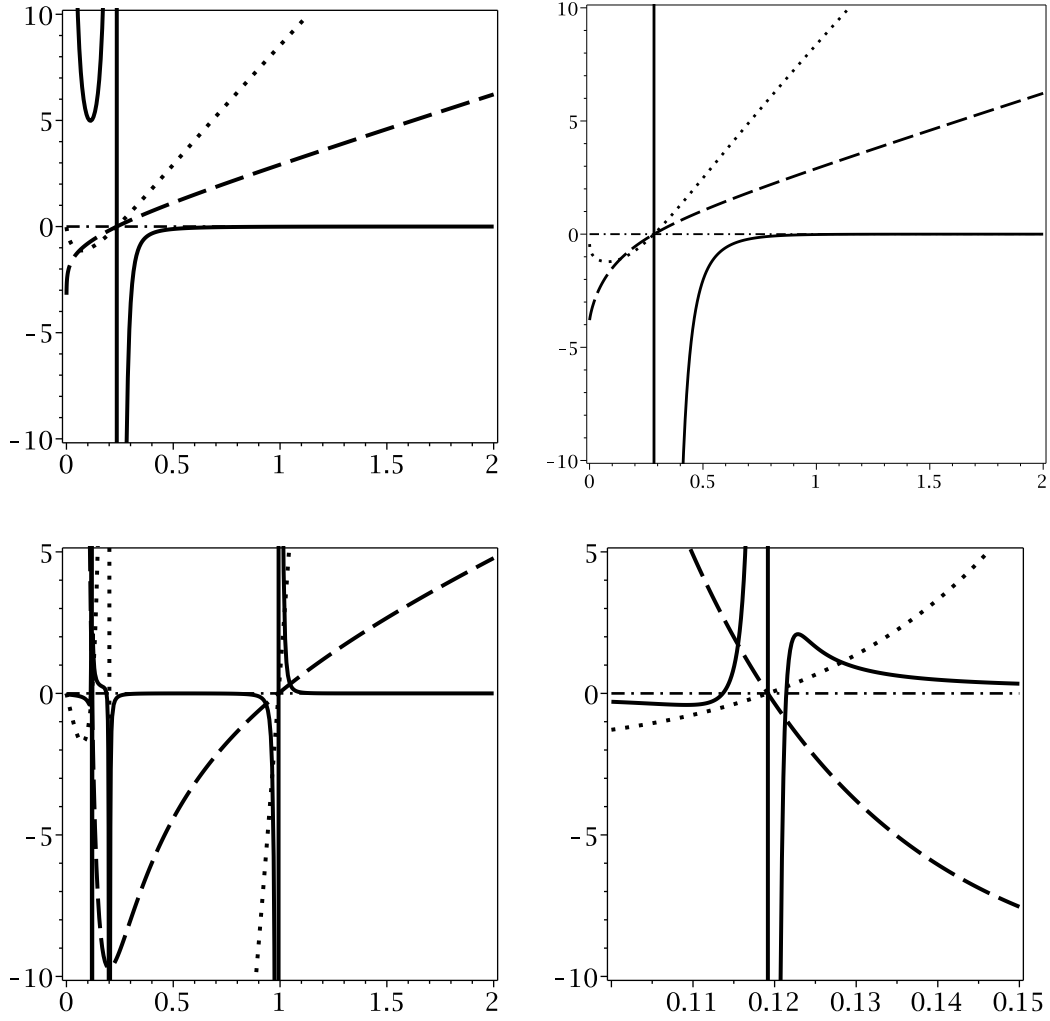


FIG. 5: HPEM Ricci scalar (solid line), heat capacity (dotted line) and temperature (dashed line) versus S for $l = 1$. **HNED model:** (up-left panel) and **SNED model:** (up-right panel): $q = 0.3$, $\beta = 1$. **CNED model:** (down-left panel) and **CNED model:** (down-right panel): $q = 1$, $\alpha = 0.007$ (different scales).

points of the heat capacity of the BTZ black holes in presence of each nonlinear models matched with divergencies of TRS of this metric.

Also, we found that in case of HNED and SNED theories, there is no divergency for heat capacity. It means that, like Maxwell theory, these two theories have no second type phase transition. These two nonlinear theories of electrodynamics, preserved the characteristic behavior of the Maxwell theory in case of heat capacity. On other hand, for CNED model, two roots and one divergence point was found for the heat capacity. In other words, due to contribution of the nonlinear electromagnetic field, heat capacity enjoys the existence of one more phase transition point of type one and a phase transition of the type two. In essence this theory is a generalization of the Maxwell theory. But this generalization added another property to heat capacity that was not observed for the Maxwell theory.

Finally it is worthwhile to mention a comment related to Legendre invariance. It was shown that [98] the Legendre invariance alone is not sufficient to guarantee a unique description of thermodynamical metrics in terms of their curvatures. In addition to Legendre invariance, one needs to demand curvature invariance under a change of representation. Therefore, it will be worthwhile to investigate both Legendre and curvature invariances. In addition, it will be interesting to think about the fundamental relation between the following two issues: (I) Agreement of thermodynamical curvature results with usual thermodynamical approaches (such as the heat capacity); (II) Curvature invariance in addition to the Legendre invariance. It is also worthwhile to probe the fundamentality of cases (I) and (II) to find that considering which one may leads to satisfy another one. Although first issue has been investigated

for special cases [98], the second one has been remained unanalyzed yet. We may address them in an independent work in the future.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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